A PREDUAL OF l_1 WHICH IS NOT ISOMORPHIC TO A $C(K)$ SPACE

BY

Y. BENYAMINI† AND J. LINDENSTRAUSS

ABSTRACT

We give an example of a Banach space X such that (i) X^* is isometric to l_1, (ii) X is isometric to a subspace of $C(\omega^\omega)$ and (iii) X is not isomorphic to a complemented subspace of any $C(K)$ space.

The example described in the abstract solves a problem which arose in several places in the literature. The first time the problem was mentioned seems to be in the paper by Bessaga and Pelczynski [1]. They give a complete isomorphic classification of all $C(K)$ spaces (= the space of all continuous functions on a compact Hausdorff K) whose duals are isometric to l_1. The authors of [1] ask whether every space whose dual is isometric (or even isomorphic) to l_1 must be isomorphic to a $C(K)$ space and thus covered by their classification. In a more general context the problem arose, e.g., in [4], [5] and [6]. In these papers much space is devoted to the study of $\mathscr{L}_\infty$ spaces. These are those Banach spaces whose finite-dimensional subspaces behave like the finite-dimensional subspaces of $C(K)$ spaces. The question raised in [4], [5], [6] whether every $\mathscr{L}_\infty$ space is isomorphic to a $C(K)$ space, can be viewed as the question whether spaces isomorphic to $C(K)$ spaces can be characterized by their local structure. The example given here gives a strong negative answer to this question and shows that the problem of isomorphic classification of separable $\mathscr{L}_\infty$ spaces (and the more restrictive class of separable preduals of $L_1(\mu)$ spaces) is wide open. The existence of such an example may be somewhat surprising in view of the known facts that preduals of $L_1(\mu)$ behave in many respects like $C(K)$ spaces (see e.g. [3] and [4]) and that the duals of

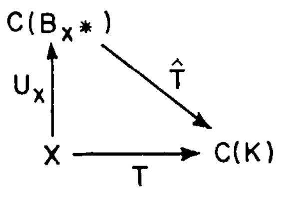
† This is a part of the first author's Ph. D. Thesis prepared in the Hebrew University of erusalem under the supervision of the second author.

separable $\mathscr{L}_\infty$ spaces are isomorphic to the duals of separable $C(K)$ spaces [7].

Let X be a Banach space. We define $\lambda(X) = \inf \| T \| \, \| T^{-1} \| \, \| P \|$ where the inf is taken over all possible isomorphisms T form X into a $C(K)$ space and all possible projections P form $C(K)$ onto TX. If X is not isomorphic to a complemented subspace of any $C(K)$ space we put $\lambda(X) = \infty$. The following trivial lemma shows that in order to compute $\lambda(X)$ we do not have to consider all possible embeddings of X into an arbitrary $C(K)$ space. It is enough to consider the canonical embedding $U_X: X \to C(B_X{}^*)$ where $B_X{}^*$ is the unit ball of X^* in the w^* topology and $U_X x(x^*) = x^*(x)$.

LEMMA. *For every Banach space* X, $\lambda(X) = \inf \| P \|$, *the infimum is taken over all bounded linear projections* P *from* $C(B_X{}^*)$ *onto* $U_X X$.

PROOF. Put $\beta(X) = \inf \| P \|$, the infimum over all P as in the statement of the lemma. Clearly $\beta(X) \geqq \lambda(X)$. To prove the converse, observe that if T is any bounded linear operator form X into a $C(K)$ space, there is an operator $\hat{T}: C(B_X{}^*) \to C(K)$ so that $\| \hat{T} \| = \| T \|$ and the following diagram commutes



A suitable $\hat{T}$ is defined by

$$\hat{T}f(k) = \| T \| f(T^* \delta_k / \| T \|)$$

where $f \in C(B_X{}^*)$, $k \in K$ and $\delta_k \in C(K)^*$ is the Dirac measure at the point k. (If $T = 0$ we take $\hat{T} = 0$.) If T is an isomorphism of X into $C(K)$ and Q a projection from $C(K)$ onto TX then $U_X T^{-1} Q \hat{T}$ is a projection from $C(B_X{}^*)$ onto $U_X X$ of norm $\leqq \| T \| \, \| T^{-1} \| \, \| Q \|$. This shows that $\lambda(X) \geqq \beta(X)$ Q.E.D.

We shall now present a construction which corresponds to a predual X of an L_1 space another predual $Y = Y(X)$ of an L_1 space with $\lambda(Y) > \lambda(X)$ (if $\lambda(X) < \infty$). By iterating this construction we get preduals Z of L_1 spaces with $\lambda(Z)$ arbitrarily large and by taking a suitable direct sum also such Z with $\lambda(Z) = \infty$. Actually we shall not work with arbitrary preduals of L_1 spaces. Our X and $Y = Y(X)$ also have an extreme point in their unit balls (i.e., both X and Y

are the spaces of all affine continuous functions on suitable Choquet simplices (cf. for example [3])).

Let $X = A(S)$ be the space of all affine continuous functions on a Choquet simplex S. Let K be the compact Hausdorff space defined as follows: As a set, K is a disjoint union of the form $\{p, q, r_1, r_2, r_3\} \cup \bigcup_{\substack{i=1,2,3,4\cdots, \\ j=1,2,3.}} S_{i,j}$ where each $S_{i,j}$ is a copy of S. (We denote by $\phi_{i,j}: S \to S_{i,j}$ the identification map from S onto $S_{i,j}$.) The topology on K is defined by requiring that each $S_{i,j}$ is closed and open in K and its topology coincides with that of S (i.e. $\phi_{i,j}$ is a homeomorphism). The points p and q are isolated points of K while a basis for the neighborhoods of r_j $(j=1,2,3)$ is given by the sets $\{r_j\} \cup \bigcup_{i=n}^{\infty} S_{i,j}$, $n = 1, 2, \cdots$. Let Y be the subspace of $C(K)$ consisting of all $f \in C(K)$ such that $f_{|S_{i,j}}$, the restriction of f to $S_{i,j}$, is affine and so that

$$f(r_1) = (f(p) + f(q))/2, \quad f(r_2) = (2f(p) + f(q))/3, \quad f(r_3) = (f(p) + 2f(q))/3.$$

(The affine structure on $S_{i,j}$ is again that induced by $\phi_{i,j}$ from S.) The norm in Y is the supremum norm.

It is easy to check that every $y^* \in Y^*$ has a unique representation of the form

$$y^* = \sum_{i,j} y^*_{i,j} + t_1\delta_p + t_2\delta q$$

with

$$\| y^* \| = \sum_{i,j} \| y^*_{i,j} \| + |t_1| + |t_2|$$

where $y^*_{i,j}(f) = y^*_{i,j}(f_{|S_{i,j}}) \in A(S_{i,j})^*$ and δ_p and δ_q are the Dirac measures ($\delta_p(f) = f(p)$). Hence

$$Y^* = R \oplus R \oplus \sum_{i,j} \oplus X^*_{i,j}$$

where every $X^*_{i,j}$ is isometric to X^*, R is the one dimensional space and all direct sums are in the l_1 sense. Thus Y^* is an $L_1(\nu)$ space for some measure ν. Moreover the function identically equal to 1 belongs to Y and therefore the unit ball of Y has an extreme point. It follows that Y can also be considered as the space of all affine continuous functions on some Choquet simplex.

The main result of this note is

THEOREM. *Let X be the space of all affine continuous functions on some Choquet simplex. Let $Y = Y(X)$ be the space constructed above. Then*

$$\lambda(Y) \geqq \lambda(X) + (500\lambda(X))^{-1}$$

PROOF. We assume that $\lambda(X) < \infty$ If $\lambda(X) = \infty$ then clearly also $\lambda(Y) = \infty$ and there is nothing to prove.

Let us first introduce some more notations. We will use, whenever it is convenient, a single index α to stand for an arbitrary pair (i,j) $i = 1,2,3,4,\cdots$, $j = 1,2,3$. For any α define the operator $J_\alpha : X \to Y$ by

$$J_\alpha f(k) = \begin{cases} f(s) & \text{if } k = \phi_\alpha s \\ 0 & \text{if } k \notin S_\alpha. \end{cases}$$

(Recall that $\phi_\alpha : S \to S_\alpha$ is the identification map). The operator J_α identifies X with the subspace of Y consisting of all these functions which vanish outside S_α. The map J_α has a one-sided inverse $R_\alpha : Y \to X$ of norm 1 defined by $R_\alpha f(s) = f(\phi_\alpha s)$.

Let $e_1 \in Y$ be the function on K defined by

$$(1) \qquad e_1(p) = 1, \quad e_1(q) = 0 \qquad e_1(k) = \begin{cases} \frac{1}{2} & \text{if } k \in S_{i,1}, \quad i = 1,2,\cdots \\ \frac{2}{3} & \text{if } k \in S_{i,2}, \quad i = 1,2,\cdots \\ \frac{1}{3} & \text{if } k \in S_{i,3}, \quad i = 1,2,\cdots \end{cases}$$

Define $e_2 \in Y$ by

$$(2) \qquad e_2 = e - e_1 \quad \text{where} \quad e(k) = 1 \quad \text{for all} \quad k \in K.$$

For each α we define an operator $T_\alpha : C(B_X{}^*) \to C(B_Y{}^*)$ by

$$(3) \qquad T_\alpha f(y^*) = \begin{cases} (1 - \psi_\alpha(y^*))f(J_\alpha^* y^*/(1 - \psi_\alpha(y^*))), & \text{if } \psi_\alpha(y^*) < 1 \\ 0 & \text{if } \psi_\alpha(y^*) = 1, \end{cases}$$

where $f \in C(B_X{}^*)$, $y^* \in B_Y{}^*$, and

$$(4) \qquad \psi_\alpha(y^*) = \left| (y^* - R_\alpha^* J_\alpha^* y^*)(e_1) \right| + \left| (y^* - R_\alpha^* J_\alpha^* y^*)(e_2) \right|.$$

The operator T_α is well defined. Indeed, by (1) and (2) we have that $\| e_1 \pm e_2 \| \leqq 1$, hence $\psi_\alpha(y^*) \leqq \| y^* - R_\alpha^* J_\alpha^* y^* \|$ and thus

$$(5) \qquad \begin{aligned} \psi_\alpha(y^*) + \| J_\alpha^* y^* \| &= \psi_\alpha(y^*) + \| R_\alpha^* J_\alpha^* y^* \| \\ &\leqq \| y^* - R_\alpha^* J_\alpha^* y^* \| + \| R_\alpha^* J_\alpha^* y^* \| = \| y^* \| \leqq 1. \end{aligned}$$

It follows that $J_\alpha^* y^*/(1 - \psi_\alpha(y^*)) \in B_X^*$. It is easily checked that $T_\alpha f(y^*)$ is a continuous function of y^*.

We note further that $\| T_\alpha \| \leqq 1$ and that the following diagram commutes:

$$(6) \qquad \begin{array}{ccc} C(B_{X^*}) & \xrightarrow{T_\alpha} & C(B_{Y^*}) \\ U_X \uparrow & & \uparrow U_Y \\ X & \xrightarrow[J_\alpha]{} & Y \end{array}$$

Indeed, if $x \in X$, $y^* \in B_Y$ with $\psi_\alpha(y^*) < 1$ then

$$\begin{aligned} T_\alpha U_X x(y^*) &= (1 - \psi_\alpha(y^*)) U_X x(J_\alpha^* y^*/(1 - \psi_\alpha(y^*))) \\ &= J_\alpha^* y^*(x) = y^*(J_\alpha x) = U_Y J_\alpha x(y^*). \end{aligned}$$

If $\psi_\alpha(y^*) = 1$ the verification is similar.

Put

$$(7) \qquad \lambda = \lambda(X), \qquad \varepsilon = 1/500\lambda.$$

We assume that there is a projection P of norm $\leqq \lambda + \varepsilon$ from $C(B_{Y^*})$ onto $U_Y Y$ and show that this leads to a contradiction.

By (6) $R_\alpha U_Y^{-1} P T_\alpha U_X$ is the identity map of X and hence by (7)

$$(8) \qquad \| R_\alpha U_Y^{-1} P T_\alpha \| \geqq \lambda.$$

Since X^* is an L_1 space and $R_\alpha e$ is an extreme point in the unit ball of X it follows from (8) that there is an $x_\alpha^* \in X^*$ so that[†]

$$(9) \qquad 1 = \| x_\alpha^* \| = x_\alpha^*(R_\alpha e), \qquad \| (R_\alpha U_Y^{-1} P T_\alpha)^* x_\alpha^* \| \geqq \lambda.$$

Put $y_\alpha^* = R_\alpha^* x_\alpha^* \in Y^*$. Then

$$(10) \qquad 1 = \| y_\alpha^* \| = y_\alpha^*(e), \qquad \| T_\alpha^*(U_Y^{-1} P)^* y_\alpha^* \| \geqq \lambda.$$

From the first part of (10) and the definition of Y it follows that

$$(11) \qquad \begin{aligned} &w^*\lim_i y_{1,i}^* = (\delta_p + \delta_q)/2 \qquad w^*\lim_i y_{i,2}^* = (2\delta_p + \delta_q)/3 \\ &w^*\lim_i y_{i,3}^* = (\delta_p + 2\delta_q)/3 \end{aligned}$$

[†] Actually this is the case only if the operator $R_\alpha U_y^{-1} P T_\alpha$ attains its norm. In the general case, we must replace λ by an arbitrary $\lambda' < \lambda$. This will require only some trivial changes in the argument.

Define now

$$(12) \qquad \mu_\alpha = (U_Y^{-1}P)^* y_\alpha^* \in C(B_{Y^*})^*.$$

Then

$$(13) \qquad \| T_\alpha^* \mu_\alpha \| \geqq \lambda, \qquad \| \mu_\alpha \| \leqq \| P^* \| \leqq \lambda + \varepsilon,$$

and

$$(14) \qquad \begin{aligned} \mu_1 &\overset{def}{=} w^*\lim_i \mu_{i,1} = ((U_Y^{-1} P)^* \delta_p + (U_Y^{-1} P)^* \delta_q)/2 \\ \mu_2 &\overset{def}{=} w^*\lim_i \mu_{i,2} = (2(U_Y^{-1}P)^*\delta_p + (U_Y^{-1}P)^*\delta_q)/3 \\ \mu_3 &\overset{def}{=} w^*\lim_i \mu_{i,3} = ((U_Y^{-1}P)^*\delta_p + 2(U_Y^{-1}P)^*\delta_q)/3 \end{aligned}$$

In particular,

$$(15) \qquad \mu_1 = (\mu_2 + \mu_3)/2.$$

Since $PU_Ye = U_Ye$,

$$(U_Y^{-1}P)^*\delta_p(U_Ye) = \delta_p(U_Y^{-1}PU_Ye) = \delta_p(e) = 1,$$

with a similar relation holding with q instead of p, we get that

$$(16) \qquad \mu_j(U_Ye) = 1 \qquad j = 1,2,3.$$

(μ_j being a linear functional on $C(B_{Y^*})$, can and will be considered also as a measure on B_{Y^*}. If we consider μ_j as a measure, we write (16) as

$$\int_{B_{Y^*}} U_Y(e)d\mu_j = 1)$$

For any index α and positive number $\eta < 1$ let $G_\alpha^\eta = \{y^*; y^* \in B_{Y^*}, \psi_\alpha(y^*) \geqq \eta\}$. Then for $f \in C(B_{X^*})$ with $\|f\| = 1$ we have by (3) that $|T_\alpha f(y^*)| < 1 - \eta$ for $y^* \in G_\alpha^\eta$. Thus for every measure μ on B_{Y^*} whose support is contained in G_α^η we get that

$$(17) \qquad \| T_\alpha^* \mu \| \leqq (1 - \eta) \| \mu \|.$$

Let us define the measures σ_α and τ_α on B_{Y^*} by

$$(18) \qquad \sigma_\alpha(A) = \mu_\alpha(A \cap G_\alpha^{10\varepsilon}), \qquad \tau_\alpha = \mu_\alpha - \sigma_\alpha$$

for every Borel set A in B_{Y^*} (ε is the number given in (7)). By (13) and (17),

$$\lambda \leqq \| T_\alpha^* \mu_\alpha \| \leqq \| T_\alpha^* \sigma_\alpha \| + \| T_\alpha^* \tau_\alpha \| \leqq (1 - 10\varepsilon) \| \sigma_\alpha \| + \| \tau_\alpha \|$$
$$= (1 - 10\varepsilon) \| \sigma_\alpha \| + \| \mu_\alpha \| - \| \sigma_\alpha \| \leqq \lambda + \varepsilon - 10\varepsilon \| \sigma_\alpha \|.$$

Hence,

(19) $$\| \sigma_\alpha \| \leqq 1/10.$$

We note also that by (18)

(20) $$\tau_\alpha \text{ is supported on } \{y^*; \psi_\alpha(y^*) \leqq 10\varepsilon\}.$$

Consider now the following three subsets of B_{Y^*}

$$F_1 = \{y^*; y^* = t(\delta_p + \delta_q)/2 + u^*, |t| + \|u^*\| \leqq 1, |u^*(e_1)| + |u^*(e_2)| \leqq 10\varepsilon\}$$
$$F_2 = \{y^*; y^* = t(2\delta_p + \delta_q)/3 + u^*, |t| + \|u^*\| \leqq 1, |u^*(e_1)| + |u^*(e_2)| \leqq 10\varepsilon\}$$
$$F_3 = \{y^*; y^* = t(\delta_p + 2\delta_q)/3 + u^*, |t| + \|u^*\| \leqq 1, |u^*(e_1| + |u^*(e_2)| \leqq 10\varepsilon\} \quad (21)$$

These sets have the following property: If, for a given j, $\psi_{i,j}(z_i^*) \leqq 10\varepsilon$ for $i = 1, 2, \cdots$ and some $z_i^* \in B_{Y^*}$ then any limit point of the sequence $\{z_i^*\}_{i=1}^\infty$ belongs to F_j. Indeed, take e.g. $j = 1$ and assume that $\psi_{i,j}^*(z_i) \leqq 10\varepsilon$. This means, by (4), that $z_i^* = u_i^* + v_i^*$ where $\|u_i^*\| + \|v_i^*\| = \|z_i^*\| \leqq 1$, $v_i^* = R_{i,1}^* J_{i,1}^* z_i^*$ and $|u_i^*(e_1)| + |u_i^*(e_2)| \leqq 10\varepsilon$. By the definition of Y every limit point of the sequence v_i^* is of the form $t\delta_{r_1} = t(\delta_p + \delta_q)/2$. Hence every limit point of $\{z_i^*\}$ belongs to F_1. It follows from this observation and (20) that every w^* limit point of $\{\tau_{i,j}\}_{i=1}^\infty$ is supported on F_j.

We note that if $y^* \in F_1 \cap F_2$ then

$$y^* = t_1(\delta_p + \delta_q)/2 + u_1^* = t_2(2\delta_p + \delta_q)/3 + u_2^*.$$

By applying the functionals to e_1 and e_2 we get that

$$|t_1/2 - 2t_2/3| < 20\varepsilon, \qquad |t_1/2 - t_2/3| < 20\varepsilon \ ,$$

and hence $|t_1|, |t_2| \leqq 120\varepsilon$. Similar computations for points in $F_1 \cap F_3$ or $F_2 \cap F_3$ show that if H_j is the subset of F_j obtained by requiring in (21) that $|t| \geqq 125\varepsilon$ then every set H_j is disjoint from the union of the two F sets with different indices.

We return now to the measures $\mu_\alpha = \sigma_\alpha + \tau_\alpha$ (cf, (18)). By (14), (19), (20) and the preceding remarks we get that for $j = 1, 2, 3$

(22) $$\mu_j = \tau_j + \sigma_j, \quad \tau_j \text{ supported on } F_j, \quad \|\sigma_j\| \leqq 1/10.$$

For every j we decompose τ_j into two measures by putting

(23) $$\rho_j(A) \leqq \tau_j(A \cap H_j), \quad \gamma_j = \tau_j - \rho_j.$$

By the definition of H_j and F_j it follows that γ_j is supported on the set $\{y^*; |y^*(e)| \leqq 150\varepsilon\}$. Hence by (7)

$$\left| \int_{B_{Y^*}} U_Y(e) d\gamma_j \right| \leqq 150\varepsilon , \quad \| \gamma_j \| \leqq 150\varepsilon(\lambda + \varepsilon) < 1/3.$$

By (22),

$$\left| \int_{B_{Y^*}} U_Y(e) d\sigma_j \right| \leqq \| \sigma_j \| \leqq 1/10.$$

Hence, by (16)

(24) $$\| \rho_j \| \geqq \left| \int_{B_{Y^*}} U_Y(e) d\rho_j \right| \geqq 1 - 1/3 - 1/10 > 1/2,$$

By (15)

$$\rho_1 = \frac{\rho_2 + \gamma_2 + \sigma_2 + \rho_3 + \gamma_3 + \sigma_3}{2} - \gamma_1 - \sigma_1,$$

but this contradicts (24) since $\rho_2, \gamma_2, \rho_3, \gamma_3$ and γ_1 all vanish on subsets of H_1, while $\| (\sigma_2 + \sigma_3)/2 - \sigma_1 \| \leqq 1/5$. This concludes the proof of the theorem.

Using the theorem and starting from any $A(S)$ space X_1 we can construct inductively a sequence of simplex spaces $X_{n+1} = Y(X_n)$ so that, if $\lambda_n = \lambda(X_n)$ then $\lambda_{n+1} \geqq \lambda_n + 1/500\lambda_n$ and hence $\lambda_n \uparrow \infty$. The direct sum in the c_0 norm $X_\infty = (\Sigma \oplus X_n)_{c_0}$ is a space whose dual is an L_1 space for which $\lambda(X_\infty) = \infty$. If we start with $X_1 =$ the one dimensional space, it is clear that X_∞ will be isometric to a subspace of $C(\omega^\omega)$ (cf. e.g. [1] for the terminology). Hence we get

COROLLARY 1. *There is a Banach space X such that*

(1) *X^* is isometric to l_1*

(2) *X is isometric to a subspace of $C(\omega^\omega)$*

(3) *X is not isomorphic to a complemented subspace of any $C(K)$ space.*

In [2] Gurari constructed a separable Banach space whose dual is an L_1 space, which has some special interesing properties. This space is unique up to almost isometry (cf. [2] and [3] for details). In [8] it was shown that every separable predual of $L_1(\mu)$ is isometric to a complemented subspace of the Gurari space. From Corollary 1 we get thus

COROLLARY 2. *The Gurari space is not isomorphic to a complemented subspace of a $C(K)$ space.*

REFERENCES

1. C. Bessaga and A. Pelczynski, *Spaces of continous functions (IV)*, Studia Math. **19** (1960), 53–62.

2. V. I. Gurari, *Space of universal disposition, isotropic spaces and the Mazur problem on rotations of Banach spaces*, Sibirski Math. Z. 7 (1966), 1002–1013.

3. A. J. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1 spaces and their representing matrices*, Acta Math. **126** (1971), 165–193.

4. J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. No. 48, 1964.

5. J. Lindenstrauss and A. Pelczynski, *Absolutely summing operators in $\mathscr{L}_p$ spaces and their applications*, Studia Math. **29** (1968), 275–326.

6. J. Lindenstrauss and H. P. Rosenthal, *The $\mathscr{L}_p$ spaces*, Israel J. Math. 7 (1969), 325–349.

7. C. Stegall, *Banach spaces whose duals contain l_1 (Γ) with applications to the study of dual L_1 (μ) spaces*, Trans. Amer. Math. Soc. (to appear).

8. P. Wojtaszczyk, *Some remarks on the Gurari space*, Studia Math. **41** (1972), 207–210.

THE HEBREW UNIVERSITY OF JERUSALEM